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Original article

Relative Entropy Model of Uncertain Random Shortest Path*

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Abstract

The shortest path problem is one of network optimization problems. This paper considers a shortest path problem under the situation where lengths of arcs in a network include both uncertainty and randomness, and focuses on the case that the lengths of arcs are expressed by uncertain random variables. This paper presents a new type of model: relative entropy model of shortest path. By the definition of relative entropy of the uncertain random variables, relative entropy model of shortest path problem is proposed to find the shortest path which fully reflects uncertain and random information. This model is formulated to find a shortest path whose chance distribution minimizes the difference from the ideal one. A numerical example is given to illustrate the model's effectiveness.

Keywords: Shortest path problem, Uncertain random variable, Chance distribution, Uncertain random network

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I. Introduction

In mathematics, the network generally refers to a weighted graph. The network is composed of nodes, arcs and arc weights. The network is called classical if arc weights are constants. In recent years network optimization has become an important research content in operational research. It includes areas such as shortest path, network flows, vehicle routing and Chinese postman problems and so on. The shortest path problem concentrates on finding a path with minimum distance, time, or cost from the source node to the destination node. It is one of the most fundamental problems in network theory. Many network optimization problems, including transportation, routing, communications, and supply chain management, can be regarded as special cases of the shortest path problem. The shortest path problem of classical network with certain arc weights have been studied intensively. And many efficient algorithms have been developed in the 50s and 60s of last century. Some representative algorithms were proposed by Bellman (1958), Dijkstra (1959), Dreyfus (1969), and Floyd (1962), which made the shortest path problem occupy a central position in a network.

However, because of failure, maintenance, or other reasons, the arc weights are indeterministic in many situations. So some researchers believed that these indeterministic phenomena conform to randomness and they introduced probability theory into the shortest path problem. If the sample size is large enough, then the estimated probability distribution may be close enough to the real frequency. As a result, we can employ the probability theory to deal with this kind of indeterminacy factor. Random network was first investigated by Frank and Hakimi (1965). After that, the random network was well developed and widely applied. And many researchers have done lots of work on stochastic shortest path problem. For example, Frank (1969), Mirchandani (1976) and Sigal et al. (1980) studied probabilistic distribution of the shortest path when the network arc of weighs are random variables. Other researchers, such as Hall (1986), Loui (1983) and Fu (1998), have also done a lot of work in this field.

Some researchers believed that it is suitable to regard these indeterministic phenomena as uncertainty and they introduced uncertainty theory into the shortest path problem. That indeterminacy cannot be described by random variables because no samples are available. So we can only employ the uncertainty theory to deal with this kind of indeterminacy factor. Uncertain network was first explored by Liu (2009a) for modeling project scheduling problem. Gao (2011) gave the inverse uncertainty distribution of the shortest path length and investigated solutions to the α-shortest path and the most shortest path in an uncertain network in 2011. Besides, the uncertainty distribution of the maximum flow problem of an uncertain network was discussed by Han et al. (2014a), the uncertain minimum cost flow problem was dealt with by Ding (2014b), and Chinese postman problem was explored by Zhang and Peng (2012) for an uncertain network.

In many real cases, uncertainty and randomness simultaneously appear in a complex network. In order to describe this phenomenon, Liu (2014c) gave the concept of the uncertain random network in which some lengths are random variables and others are uncertain variables. At the same time, Liu (2014c) studied chance distribution of the shortest path in an uncertain random network. After that, Sheng and Gao (2014d) proposed two models for the shortest path problem of an uncertain random network. Sheng et al. (2014e) gave an ideal chance distribution of the minimum spanning tree of an uncertain random network, proposed an area chance distribution model, and designed an algorithm to find the minimum spanning tree of uncertain random network. Sheng and Shi (2015a)(2015b) proposed a distance chance distribution model and a relative entropy chance distribution model to find the minimum spanning tree of uncertain random network. Sheng and Gao (2014d) studied the chance distribution of the maximum flow of an uncertain random network.

This paper will further study the shortest path problem under the framework of chance theory. The remainder of this paper is organized as follows. In Section 2, some basic concepts and properties of chance theory used throughout this paper are introduced. In Section 3, the uncertain random shortest path problem is described. Section 4 gives a definition of the shortest path about relative entropy of chance distribution and proposes a model of shortest path in an uncertain random network. In Section 5, a numerical example is given to illustrate the conclusions presented in Sections 4. Section 6 gives a brief summary to this paper and discusses future studies.

II. Preliminaries

In this section, we first review some concepts and theorems of uncertainty theory and chance theory.

2.1. Uncertainty Theory

In order to deal with belief degrees, Liu (2007) founded uncertainty theory in and refined it (2010). Uncertain measure is a type of set function used to indicate the belief degree that an uncertain event may occur. Furthermore, uncertain measure was also actively investigated by Gao (2009c). As a fundamental concept in uncertainty theory, uncertain variable was presented by Liu (2007). Liu (2010) also proposed the concept of uncertainty distribution, inverse uncertainty distribution. Later, Peng and Iwamura (2010a) derived a sufficient and necessary condition for uncertainty distribution. In addition, the concept of independence was proposed by Liu (2009b). Liu (2010) presented the operational law of uncertain variables. In order to rank uncertain variables, Liu (2010) proposed the concepts of expected value, variance and moments of uncertain variables. Many researchers have contributed significantly in this area. The linearity of expected value operator was verified by Liu (2010). As an important contribution, Liu and Ha (2010b) developed a formula for calculating the expected values of monotone functions of uncertain variables. Up to now, uncertainty theory has become a branch of axiomatic for modeling belief degrees. Uncertainty theory and its applications have experienced explosive growth. Liu (2009b) first proposed uncertain programming theory to model uncertain optimization problems. Liu (2010) studied uncertain statistics, which is a methodology for collecting and interpreting expert's experimental data by uncertainty theory and so on.

Definition 1 *(Liu (2007)) Let* L *be a* σ *-algebra on a nonempty set* Γ. *A set function* $M: L \to [0,1]$ *is called an uncertain measure if it satisfies the following axioms:*

Axiom 1: (Normality Axiom) $M\{\Gamma\}=1$ for the universal set Γ.

Axiom 2: (Duality Axiom) $M\{\Lambda\}+M\{\Lambda^c\}=1$ for any event Λ .

Axiom 3: (Subadditivity Axiom) For every countable sequence of events $\Lambda_1, \Lambda_2, \cdots,$ we have $M\left\{\bigcup_{i=1}^{\infty}\Lambda_i\right\} \le \sum_{i=1}^{\infty} M\{\Lambda_i\}$.

Besides, the product uncertain measure on the product σ *-algebra* L is defined by the following product axiom.

Axiom 4: (Product Axiom) (Liu (2009b)) Let (Γ_k, L_k, M_k) *be uncertainty spaces for* $k=1,2,\cdots$ *The product uncertain measure* M *is an uncertain measure satisfying* $M\left\{\prod_{k=1}^{\infty}\Lambda_k\right\} = \bigwedge_{k=1}^{\infty} M_k\{\Lambda_k\}$ *where* Λ_k *are arbitrarily chosen events from* L_k *for* $k=1,2,\cdots$, *respectively.*

An uncertain variable is essentially a measurable function from an uncertainty space to the set of real numbers. In order to describe an uncertain variable, a concept of uncertainty distribution is defined as follows.

Definition 2 *(Liu (2007)) The uncertainty distribution of an uncertain variable* ξ *is defined by* $\Phi(x) = M\{\xi \le x\}$ *for any* $x \in \Re$.

Definition 3 *(Liu (2009b)) The uncertain variables* $\xi_1, \xi_2, \cdots, \xi_n$ *are said to be independent if* $M\left\{\bigcap_{i=1}^{n}(\xi_i \in B_i)\right\} = \bigwedge_{i=1}^{n} M\{\xi_i \in B_i\}$ *for any Borel sets* $B_1, B_2, \cdots, B_n$.

Let ξ be an uncertain variable with regular uncertainty distribution Φ . Then the inverse function Φ^{-1} is called the inverse uncertainty distribution of ξ .

The distribution of a monotonous function of uncertain variables can be obtained by the following theorem.

Theorem 1 *(Liu (2010)) Let* $\xi_1, \xi_2, \cdots, \xi_n$ *be independent uncertain variables with uncertainty distributions* $\Phi_1, \Phi_2, \cdots, \Phi_n$ *, respectively. If* $f(\xi_1, \xi_2, \cdots, \xi_n)$ *is strictly increasing with respect to* $\xi_1, \xi_2, \cdots, \xi_m$ *and strictly decreasing with respect to* $\xi_{m+1}, \xi_{m+2}, \cdots, \xi_n$ *, then* $\xi = f(\xi_1, \xi_2, \cdots, \xi_n)$ *is an uncertain variable with an inverse uncertainty distribution*

$$\Phi^{-1}(\alpha) = f(\Phi_1^{-1}(\alpha), \Phi_2^{-1}(\alpha) \cdots, \Phi_m^{-1}(\alpha), \Phi_{m+1}^{-1}(1-\alpha), \cdots, \Phi_n^{-1}(1-\alpha)).$$

Theorem 2 *(Liu (2010)) Let* $\xi_1, \xi_2, \cdots, \xi_n$ *be independent uncertain variables with uncertainty distributions* $\Phi_1, \Phi_2, \cdots, \Phi_n$, *respectively. If the function* $f(x_1, x_2, \cdots, x_n)$ *is strictly increasing with respect to* $x_1, x_2, \cdots, x_m$ *and strictly decreasing with respect to* $x_{m+1}, x_{m+2}, \cdots, x_n$ *, then* $\xi = f(\xi_1, \xi_2, \cdots, \xi_n)$ *is an uncertain variable with uncertainty distribution*

$$\Phi(x) = \sup_{f(x_1, x_2, \cdots, x_n)=x} \left(\min_{1 \le i \le m} \Phi_i(x_i) \wedge \min_{m+1 \le i \le n} (1 - \Phi_i(x_i))\right).$$

2.2. Chance Theory

In 2013, Liu (2013) first proposed chance theory via the concepts of uncertain random variable and chance measure in order to describe the situation that uncertainty and randomness appear in a system. Some related concepts of uncertain random variables such as chance distribution, expected value, variance were also presented by Liu (2013). As an important contribution to chance theory, Liu (2013a) presented an operational law of uncertain random variables. After that, uncertain random variables were discussed widely. Guo and Wang (2014f) proved a formula for calculating the variance of uncertain random variables based on uncertainty distribution. Sheng and Yao (2014g) proposed a formula to calculate the variance using chance distribution and inverse chance distribution. Sheng al et. (2015b) presented the concept of entropy of uncertain random variable.

The chance space is refer to the product $(\Gamma, L, M) \times (\Omega, A, \Pr)$, in which (Γ, L, M) is an uncertainty space and $(\Omega, A, \Pr)$ is a probability space.

Definition 4 *(Liu (2013)) Let* $(\Gamma, L, M) \times (\Omega, A, \Pr)$ *be a chance space, and let* $\Theta \in L \times A$ *be an uncertain random event. Then the chance measure of* Θ *is defined as* $\mathrm{Ch}\{\Theta\} = \int_0^1 \Pr\{\omega \in \Omega \mid M\{\gamma \in \Gamma \mid (\gamma, \omega) \in \Theta\} \ge r\} \mathrm{d}r.$

Liu (2013) proved that a chance measure satisfies normality, duality, and monotonicity properties. Besides, Hou (2014h) proved the subadditivity of chance measure.

Theoretically, an uncertain random variable is a measurable function on the chance space. It is usually used to deal with measurable functions of uncertain variables and random variables.

In order to describe an uncertain random variable, a concept of chance distribution is defined as follows.

Definition 5 *(Liu (2013a)) Let* ξ *be an uncertain random variable. Then its chance distribution is defined by* $\Phi(x) = \mathrm{Ch}\{\xi \le x\}$ *for any* $x \in \Re$.

Theorem 3 *(Liu (2013a)) Let* $\eta_1, \eta_2, \cdots, \eta_m$ *be independent random variables with probability distributions* $\Psi_1, \Psi_2, \cdots, \Psi_m$, *respectively, and let* $\tau_1, \tau_2, \cdots, \tau_n$ *be uncertain variables. Then the uncertain random variable* $\xi = f(\eta_1, \eta_2, \cdots, \eta_m, \tau_1, \tau_2, \cdots, \tau_n)$ *has a chance distribution* $\Phi(x) = \int_{\Re^m} F(x, y_1, \cdots, y_m) \mathrm{d}\Psi_1(y_1) \cdots \mathrm{d}\Psi_m(y_m)$ *where* $F(x, y_1, \cdots, y_m)$ *is the uncertainty* *distribution of uncertain variable* $f(y_1, y_2, \cdots, y_m, \tau_1, \tau_2, \cdots, \tau_n)$ *for any real numbers* $y_1, y_2, \cdots, y_m$.

Definition 6 *(Sheng* et al. *(2015b)) Let* ξ *and* η *be two uncertain random variables with chance distributions* $\Phi_\xi(x)$ *and* $\Phi_\eta(x)$, *respectively. Then the relative entropy of* ξ *from* η *is defined by* $R[\xi;\eta] = \int_{-\infty}^{+\infty} C\left(\Phi_\xi(x), \Phi_\eta(x)\right)\mathrm{d}x$ *where* $C(s,t) = s\ln\frac{s}{t} - (1-s)\ln\frac{1-s}{1-t}, 0 \le s \le 1, 0 \le t \le 1.$

Theorem 4 *(Liu (2014c)) For the network* (N, U, R, W), *assume the uncertain weights* w_{ij} *have uncertainty distributions* Υ_{ij} *for* $(i, j) \in U$, *and the random weights* w_{ij} *have probability distributions* Ψ_{ij} *for* $(i, j) \in R$, *respectively. Then the shortest path from a source node to a destination node has an ideal chance distribution*

$$\Phi(x) = \int_0^{+\infty} \cdots \int_0^{+\infty} F(x; y_{ij}, (i, j) \in R) \prod_{(i,j) \in Rs} \mathrm{d}\Psi_{ij}(y_{ij})$$

where $F(x; y_{ij}, (i, j) \in R)$ is determined by its inverse uncertainty distribution $F^{-1}(\alpha; y_{ij}, (i, j) \in R) = f(c_{ij}, (i, j) \in U \cup R)$,

$$c_{ij} = \begin{cases} \Upsilon_{ij}^{-1}(\alpha), & if \quad (i, j) \in U \\ y_{ij}, & if \quad (i, j) \in R \end{cases} \tag{1}$$

and f may be calculated by the Dijkstra algorithm for each given α.

Theorem 5 *(Sheng and Gao (2015b)) Let* $\{x_{ij}, (i, j) \in U \cup R\}$ *be a path from the source node to the destination node of an uncertain random network* (N, U, R, W) *and let* $\Psi(x)$ *be the chance distribution of the path. Then we have* $\Psi(x) \le \Phi(x), \forall x \in \Re$ *where* $\Phi(x)$ *is the ideal chance distribution of shortest path of uncertain random network.*

III. Shortest path problem

We assume the uncertain random network is always of order *n*, and has a collection of nodes, $N = \{1, 2, \cdots, n\}$. We defined two collections of arcs, $U = \{(i, j) \mid (i, j)$ are uncertain arcs$\}$, $R = \{(i, j) \mid (i, j)$ are random arcs$\}$.

Note that all deterministic arcs are regarded as special uncertain ones. Let w_{ij} denote the weights of arcs (i, j), $(i, j) \in U \cup R$, respectively. Then w_{ij} are uncertain variables if $(i, j) \in U$, and w_{ij} are random variables if $(i, j) \in R$. Write $W = \{w_{ij} \mid (i, j) \in U \cup R\}$.

Definition 7 *(Liu (2014c)) Assume* N *is the collection of nodes,* U *is the collection of uncertain arcs,* R *is the collection of random arcs, and* W *is the collection of uncertain and random weights. Then the quartette* (N, U, R, W) *is said to be an uncertain random network.*

In this paper, the nondeterministic factor is only the length of each arc. Some assumptions are listed as follows.

(1) There is only one source node and only one destination node and no cycle in each network.

(2) The weight of each arc (i, j) is a positive uncertain variable or a positive random variables, *i.e.*, the weight of a path from source node to destination node is a positive uncertain random variable.

(3) All the uncertain variables and the random variables are independent.

Assume ξ_{ij} are the lengths of arcs $(i, j)\in U\cup R$, denote $\xi=\{\xi_{ij}\,|\,(i, j)\in U\cup R\}$. We can represent the uncertain random network with uncertain random arc length as (N, U, R, ξ). Its shortest path length is a function of ξ. Obviously, this function is an uncertain random variable. A path P from a source node to destination node is represented by $\{x_{ij}\,|\,(i, j)\in U\cup R\}$ where $x_{ij}=0$ represent arc (i, j) is not in the path and $x_{ij}=1$ represent arc (i, j) is in the path. Then the weight of the path $\{x_{ij}\,|\,(i, j)\in U\cup R\}$ is $\sum_{(i,j)\in U\cup R} x_{ij}\xi_{ij}$. It is clear that $\sum_{(i,j)\in U\cup R} x_{ij}\xi_{ij}$ is just an uncertain random variable.

It has been proved that $\{x_{ij}\,|\,(i, j)\in U\cup R\}$ is a path from source node 1 to destination node n if and only if

$$\begin{cases} \displaystyle\sum_{j:(i,j)\in U\cup R} x_{ij} - \sum_{j:(j,i)\in U\cup R} x_{ji} = \begin{cases} 1, & i=1 \\ 0, & 2\le i\le n-1 \\ -1, & i=n \end{cases} \\ x_{ij}=\{0,1\}, \quad \forall (i,j)\in U\cup R. \end{cases} \tag{2}$$

IV. Relative entropy model of shortest path

By the definition of relative entropy of uncertain random variable, in order to find the optimal path of an uncertain random network (N, U, R, W), we give a definition of the shortest path of an uncertain random network. In this section, we devise a model of shortest path by relative entropy of chance distribution and ideal chance distribution. This is a distinct contribution from other optimization methods.

Definition 8 *A path P^* is called the shortest path about relative entropy chance distribution if $\int_{-\infty}^{+\infty} C\left(\Phi(x), \Phi_{P^*}(x)\right)\mathrm{d}x$ is minimum for any real number x, where $\Phi_{P^*}(x)$ is the chance distribution of path P^*, $\Phi(x)$ is the ideal chance distribution of the shortest path and* $C(s,t)=s\ln\dfrac{s}{t}-(1-s)\ln\dfrac{1-s}{1-t}, 0\le s\le 1, 0\le t\le 1.$

According to Definition 8, we minimize relative entropy between the ideal chance distribution of shortest path and chance distribution of for any path P, *i.e.*, the object function is denoted by

$\int_{-\infty}^{+\infty} C\left(\Phi(x), \Phi_P(x \mid x_{ij}, (i,j) \in U \cup R)\right) \mathrm{d}x$ and the shortest path can be obtained from the following model (3):

$$\begin{cases} \min\limits_{\{x_{ij},(i,j)\in U\cup R\}} \int_{-\infty}^{+\infty} C\left(\Phi(x), \Phi_P(x \mid x_{ij}, (i,j) \in U \cup R)\right) \mathrm{d}x \\ s.t. \\ \quad \sum\limits_{j:(i,j)\in U\cup R} x_{ij} - \sum\limits_{j:(j,i)\in U\cup R} x_{ji} = \begin{cases} 1, & i = 1 \\ 0, & 2 \le i \le n-1 \\ -1, & i = n \end{cases} \\ \quad x_{ij} = \{0,1\}, \quad \forall (i,j) \in U \cup R \end{cases} \tag{3}$$

where $C(s,t) = s\ln\dfrac{s}{t} - (1-s)\ln\dfrac{1-s}{1-t}, 0 \le s \le 1, 0 \le t \le 1,$ $\Phi(x)$ is the ideal chance distribution of the shortest path, and $\Phi_P(x \mid x_{ij}, (i,j) \in U \cup R)$ is the chance distribution of for the path $\{x_{ij}, (i,j) \in U \cup R\}$ of uncertain random network.

Theorem 6 *The model (3) can be reformulated as the following model:*

$$\begin{cases} \min\limits_{\{x_{ij},(i,j)\in U\cup R\}} \int_{-\infty}^{+\infty} C\left(\Phi(x), \int_{-\infty}^{+\infty} \Upsilon\left(x - \sum\limits_{(i,j)\in R} x_{ij} y_{ij}\right) \mathrm{d}\Psi(y)\right) \mathrm{d}x \\ s.t. \\ \quad \sum\limits_{j:(i,j)\in U\cup R} x_{ij} - \sum\limits_{j:(j,i)\in U\cup R} x_{ji} = \begin{cases} 1, & i = 1 \\ 0, & 2 \le i \le n-1 \\ -1, & i = n \end{cases} \\ \quad x_{ij} = \{0,1\}, \quad \forall (i,j) \in U \cup R \end{cases} \tag{4}$$

where $\Phi(x)$, $\Psi(y)$, $\Upsilon(r)$ and $C(s,t)$ can be obtained by $\Phi(x) = \int_0^{+\infty} \cdots \int_0^{+\infty} F(x; y_{ij}, (i,j) \in R) \prod_{(i,j)\in R} \mathrm{d}\Psi_{ij}(y_{ij})$,
$\Upsilon(r) = \sup_{\sum_{(i,j)\in U} x_{ij} r_{ij} = r} \min_{(i,j)\in U} \Upsilon_{ij}(x_{ij} r_{ij})$, $\quad \Psi(y) = \int_{\sum_{(i,j)\in R} x_{ij} y_{ij} \le y} \prod_{(i,j)\in R} \mathrm{d}\Psi_{ij}(x_{ij} y_{ij})$ and

$C(s,t) = s\ln\dfrac{s}{t} - (1-s)\ln\dfrac{1-s}{1-t}, 0 \le s \le 1, 0 \le t \le 1.$

Proof : Assume that $\{x_{ij}, (i,j) \in U \cup R\}$ is a path from source note 1 to destination note n. Let $\tau_{ij}, (i,j) \in U$ denote the lengths of uncertain arcs with regular uncertainty distributions Υ_{ij} and $\eta_{ij}, (i,j) \in R$ denote the lengths of random arcs with probability distributions Ψ_{ij}. Denote the probability distribution of $\sum_{(i,j)\in\Re} x_{ij}\eta_{ij}$ by $\Psi(y)$. From the assumption (3), we have

$$\Psi(y)=\int_{\sum_{(i,j)\in R} x_{ij}y_{ij}\le y}\prod_{(i,j)\in R}\mathrm{d}\Psi_{ij}(x_{ij}y_{ij}). \tag{5}$$

It follows from Theorem 3 that

$$\begin{aligned}\Phi_P(x\mid x_{ij},(i,j)\in U\cup R)&=\mathrm{Ch}\left\{\sum_{(i,j)\in U}x_{ij}\tau_{ij}+\sum_{(i,j)\in R}x_{ij}\xi_{ij}\le x\right\}=\int_{\Re}M\left\{\sum_{(i,j)\in U}x_{ij}\tau_{ij}+y\le x\right\}\Psi(y)\\&=\int_{\Re}M\left\{\sum_{(i,j)\in U}x_{ij}\tau_{ij}\le x-y\right\}\Psi(y).\end{aligned} \tag{6}$$

Further, it follows from Theorem 2 that

$$M\left\{\sum_{(i,j)\in U}x_{ij}\tau_{ij}\le x-y\right\}=\sup_{\sum_{(i,j)\in U}x_{ij}r_{ij}=x-y}\ \min_{(i,j)\in U}\Upsilon_{ij}(x_{ij}r_{ij})=\Upsilon\left(x-\sum_{(i,j)\in R}x_{ij}y_{ij}\right). \tag{7}$$

So we have

$$\Phi_P(x\mid x_{ij},(i,j)\in U\cup R)=\int_{-\infty}^{+\infty}\Upsilon\left(x-\sum_{(i,j)\in R}x_{ij}y_{ij}\right)\mathrm{d}\Psi(y). \tag{8}$$

By Theorem 4, the ideal chance distribution of shortest path is

$$\Phi(x)=\int_0^{+\infty}\cdots\int_0^{+\infty}F(x;y_{ij},(i,j)\in R)\prod_{(i,j)\in R}\mathrm{d}\Psi_{ij}(y_{ij}). \tag{9}$$

The theorem is proved by substituting Equations (8) and (9) into the object function of Model (3).

V. A numerical example

In this section, we will give an example to illustrate the conclusions presented above. Generally speaking, this model is difficult to solve using traditional methods. In order to solve the above model, we develop an algorithm. For example, an uncertain random network (N, U, R, W) with 5 nodes and 6 arcs is represented by Figure 1.

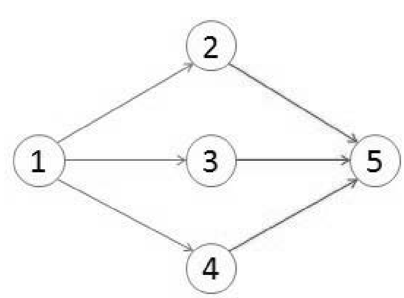

Figure 1: An uncertain random network (N, U, R, W)

Assume the uncertain weights τ_{25}, τ_{35}, τ_{45} have regular uncertainty distributions Υ_{25}, Υ_{35}, Υ_{45} and the random weights ξ_{12}, ξ_{13}, ξ_{14} have probability distributions Ψ_{12}, Ψ_{13}, Ψ_{14}, respectively.

In this paper, we assume the ideal chance distribution of shortest path is $\Phi(x)$ and the chance distribution of any path is $\Phi_P(x)$. We will solve the following problems:

(1) the ideal chance distribution of shortest path $\Phi(x)$;

(2) the chance distribution of all paths $\Phi_P(x)$;

(3) the relative entropy of ideal chance distribution and chance distribution of any path P;

(4) the shortest path of an uncertain random network.

The ideal chance distribution the shortest path of the network (N, U, R, W) is

$$\Phi(x) = \int_0^{+\infty}\int_0^{+\infty}\int_0^{+\infty} F(x; y_{12}, y_{13}, y_{14}) \mathrm{d}\Psi_{12}(y_{12})\mathrm{d}\Psi_{13}(y_{13})\mathrm{d}\Psi_{14}(y_{14}) \tag{10}$$

where $F(x; y_{12}, y_{13}, y_{14})$ is determined by its inverse uncertainty distribution $F^{-1}(\alpha; y_{12}, y_{13}, y_{14}) = (y_{12} + \Upsilon_{25}^{-1}(\alpha)) \wedge (y_{13} + \Upsilon_{35}^{-1}(\alpha)) \wedge (y_{14} + \Upsilon_{45}^{-1}(\alpha)).$

There are three paths, we denote Path 1 ($1 \to 2 \to 5$), Path 2 ($1 \to 3 \to 5$) and Path 3 ($1 \to 4 \to 5$). Then we have that the chance distribution of the Path 1 of the network (N, U, R, W) is $\Phi_{Path1}(x) = \int_0^{+\infty} F(x; y_{12}) \mathrm{d}\Psi_{12}(y_{12})$ where $F(x; y_{12})$ is determined by its inverse uncertainty distribution $F^{-1}(\alpha; y_{12}) = y_{12} + \Upsilon_{25}^{-1}(\alpha).$

Then we have that the chance distribution of the Path 2 of the network (N, U, R, W) is $\Phi_{Path2}(x) = \int_0^{+\infty} F(x; y_{13}) \mathrm{d}\Psi_{13}(y_{13})$ where $F(x; y_{13})$ is determined by its inverse uncertainty distribution $F^{-1}(\alpha; y_{13}) = y_{13} + \Upsilon_{35}^{-1}(\alpha).$

Then we have that the chance distribution of the Path 3 of the network (N, U, R, W) is $\Phi_{Path3}(x) = \int_0^{+\infty} F(x; y_{14}) \mathrm{d}\Psi_{14}(y_{14})$ where $F(x; y_{14})$ is determined by its inverse uncertainty distribution $F^{-1}(\alpha; y_{14}) = y_{14} + \Upsilon_{45}^{-1}(\alpha).$

Here, we assume τ_{25}: $L(2,5)$, τ_{35}: $L(3,4)$, τ_{45}: $L(2,4)$. Then $\Upsilon_{25}^{-1}(\alpha) = 2 + 3\alpha$, $\Upsilon_{35}^{-1}(\alpha) = 3 + \alpha$, $\Upsilon_{45}^{-1}(\alpha) = 2 + 2\alpha$. Assume ξ_{12}: $U(1,3)$, ξ_{13}: $U(2,3)$, ξ_{14}: $U(3,4)$. Then

$$\Psi_{12}(y_{12}) = \begin{cases} 0, & if \quad y_{12} \le 1 \\ \dfrac{y_{12} - 1}{2}, & if \quad 1 \le y_{12} \le 3 \\ 1, & if \quad y_{12} \ge 3, \end{cases} \tag{11}$$

$$\Psi_{13}(y_{13}) = \begin{cases} 0, & if \quad y_{13} \le 2 \\ y_{13} - 2, & if \quad 2 \le y_{13} \le 3 \\ 1, & if \quad y_{13} \ge 3, \end{cases} \tag{12}$$

and

$$\Psi_{14}(y_{14}) = \begin{cases} 0, & if \quad y_{14} \le 3 \\ y_{14} - 3, & if \quad 3 \le y_{14} \le 4 \\ 1, & if \quad y_{14} \ge 4. \end{cases} \tag{13}$$

So we obtain the ideal chance distribution $\Phi(x) = \frac{1}{2}\int_1^3\int_2^3\int_3^4 F(x; y_{12}, y_{13}, y_{14})\mathrm{d}y_{12}\mathrm{d}y_{13}\mathrm{d}y_{14}$ where $F(x; y_{12}, y_{13}, y_{14})$ is determined by its inverse uncertainty distribution $F^{-1}(\alpha; y_{12}, y_{13}, y_{14}) = (y_{12} + 2 + 3\alpha) \wedge (y_{13} + 3 + \alpha) \wedge (y_{14} + 2 + 2\alpha)$ for each given α .

So we obtain the chance distribution of Path 1 $\Phi_{Path1}(x) = \frac{1}{2}\int_1^3 F(x; y_{12})\mathrm{d}y_{12}$ where $F(x; y_{12})$ is determined by its inverse uncertainty distribution $F^{-1}(\alpha; y_{12}) = y_{12} + 2 + 3\alpha$ for each given α .

So we obtain the chance distribution of Path 2 $\Phi_{Path2}(x) = \int_2^3 F(x; y_{13})\mathrm{d}y_{13}$ where $F(x; y_{13})$ is determined by its inverse uncertainty distribution $F^{-1}(\alpha; y_{13}) = y_{13} + (3 + \alpha)$ for each given α .

So we obtain the chance distribution of Path 3 $\Phi_{Path3}(x) = \int_3^4 F(x; y_{14})\mathrm{d}y_{14}$ where $F(x; y_{14})$ is determined by its inverse uncertainty distribution $F^{-1}(\alpha; y_{14}) = y_{14} + 2 + 2\alpha$ for each given α .

By Theorem 4 and Theorem 3, we can obtain ideal chance distribution and chance distribution of each path shown in Figure 2. We can calculate relative entropies of $\Phi(x)$ and $\Phi_{Path1}(x)$, $\Phi(x)$ and $\Phi_{Path2}(x)$ and $\Phi(x)$ and $\Phi_{Path3}(x)$. They are

$$\int_{-\infty}^{+\infty} C\left(\Phi(x), \Phi_{Path1}(x)\right)\mathrm{d}x = 0.22, \qquad \int_{-\infty}^{+\infty} C\left(\Phi(x), \Phi_{Path2}(x)\right)\mathrm{d}x = 1.35,$$

$$\int_{-\infty}^{+\infty} C\left(\Phi(x), \Phi_{Path3}(x)\right)\mathrm{d}x = 2.34.$$

Lastly, we can find the shortest path is Path 1 ($1 \to 2 \to 5$) of the uncertain random network (N, U, R, W).

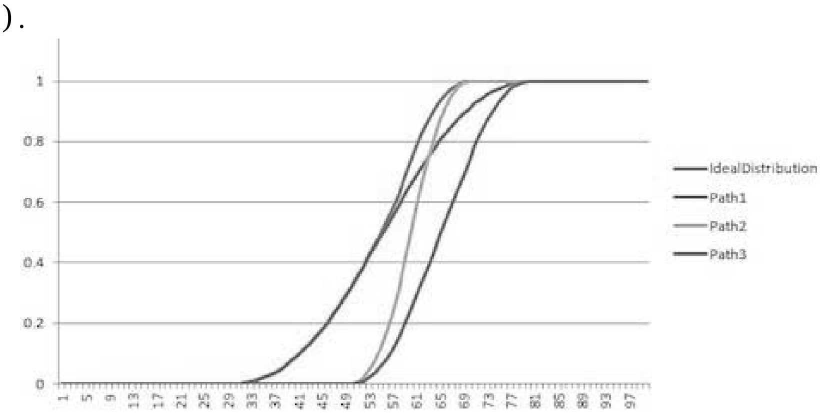

Figure 2: The ideal chance distribution and distributions Paths 1, 2, 3

For large uncertain random networks, the efficiency may be low. Some efficient algorithms may be designed to improve the efficiency in real applications.

VI. A numerical example

Chance theory provides a new approach to deal with indeterministic factors in a network with not only uncertainty but also randomness. This paper proposed a model and an algorithm to obtain the shortest path for an uncertain random network. This model was called relative entropy optimization model about the uncertain random network. At last, we gave an example to illustrate the conclusions presented.

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