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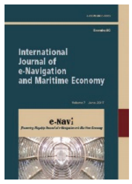

Original article

Evaluating Wave Random Path Using Multilevel Monte Carlo ☆

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Abstract

Wind waves are important due to their high energy and impact on marine activities. This phenomenon is affects directly or indirectly the construction of coastal infrastructure, shipping and recreational activities. Due to the issues presented, marine parameters are very important. In this study, we try to pay attention to wave as one of the most important marine parameters. As the movements of waves have high uncertainty, financial models can be used to simulate the wave's paths. We use the Monte Carlo method for this purpose. The Monte Carlo simulation is a flexible and simple tool that is widely used in the evaluation of random paths. To compute a random path, we require an integral discretization. In this paper, we study the valuation of European options using Monte Carlo simulation and then compare this result with multi-level Monte Carlo approach and other antithetic variables. Then, we use the multi-level Monte Carlo approach proposed by (M. B. Giles 2008) for pricing under the two-factor stochastic volatility model. We show that the multi-level Monte Carlo method reduces the computational complexity and also cost of the two-factor stochastic volatility model when compared with the standard Monte Carlo method. Also, we compare the multi-level Monte Carlo method and standard Monte Carlo method using an Euler discretization scheme and then, analyze the numerical results.

Keywords: antithetic variables technique, computational complexity, Monte Carlo simulation, multi level Monte Carlo, multi-factor stochastic volatility

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1. Introduction

The effects of water waves in coastal engineering and its impact on marine structures is very important. The waves are the most important factor in determining the geometry and composition of the coast and they also have a major impact on the design of ports, waterways and on coastal marine work. Information about the characteristics of waves such as their height and time cycle are some of the basic needs for scientists, researchers, Civil, Marine Science and Technology and Navier and fisheries Engineers and also coastal area residents. The information is important for the design, construction, implementation, installation and transfer of marine structures such as platforms, buoys, breakwaters. It is also key in estimating the amount of marine sediment to transport and also the erosion and deposition of estimates made in the vicinity of offshore structures and ports. Given that waves are generated by the wind on the water's surface, they have different heights and durations. Also, due to the impact of various factors with regard to height and the duration the waves that reach out to the offshore structures, is almost impossible to predict the parameters accurately. Therefore efforts to predict the formation of waves based on weather information and some measured variables affecting them in the desired location have to be made. Due to the random motion of waves, stochastic differential equations follow from the Brownian motion (Wiener process) Which are very similar to the behavior of stock in the financial markets.

The Monte Carlo methods are applied to calculate the expected value of a random variable. Specially, when the random variable of interest in a given stochastic differential equation, using Monte Carlo methods is more complicated when compared to other deterministic numerical methods, but they offer a better solution. Bearing in mind that financial models often require a computational complex solution, Monte Carlo methods usually enable an efficient solution and are widely used to solve them. The reason is that Monte Carlo methods are used to approximate the expected value of the sample mean. In fact, the sample mean obtained works as an unbiased estimator. Still, this method of course leads to a statistical error. Confidence interval obtained using the Monte Carlo method includes the standard deviation and inverse square root of the estimator. There are two ways to reduce the error in the standard Monte Carlo approach. First one is to increase the number of simulations and the second one is to reduce the variance of the estimator. The first approach involves squaring the number of simulations and that is computationally expensive. Thus, the second one is usually used for this reason To reduce the variance and accordingly reduce the computational complexity, various approaches have been proposed by (Glasserman 2004). These methods are known as variance reduction techniques. Some of these techniques are based on building more sophisticated estimators and others are based on enhanced random number generators. Here, we will discuss techniques based on building more sophisticated estimators, such as control variates and antithetic variables. .

Recently, the work of (Heinrich 2001), and (Kebaier 2005) suggest it as a statistical method and Romberg introduced two levels. This idea was further extended by Giles (M. B. Giles 2008) and the so-called multi-level Monte Carlo (MLMC) method was presented, which allows one to significantly accelerate the Monte Carlo classic method and approach.

This method is typically expressed by a multi-level control variate technique. By choosing the time step sizes and the number of simulations for each level, the complexity of computation is reduced when compared to the crude Monte Carlo approach. In other words, to calculate the expected value with an accuracy of $O(\mathcal{E})$, the computational cost of $O(\mathcal{E}^{-3})$ is reduced to $O(\mathcal{E}^{-2}(\log \mathcal{E})^2)$. (M. B. Giles 2008).

In this paper, we first improve the option prices under the geometric Brownian motion. To implement this, we use the multi-level Monte Carlo method and antithetic variates as a variance reduction technique. Pei Yuen Lee (2011) studied the multi-level Monte Carlo simulation for pricing options in terms of efficiency. He implemented the method using the Euler and the Milstein schemes. Dominik Huth (2012) presented multi-level Monte Carlo results based on Asian, Barrier, and American options in the Black-Scholes model.

Then, we studied the multi-level Monte Carlo method for the multi-level stochastic volatility models that are of a particular type. Stochastic volatility models are used, to model the stochastic volatility smile (smirk) that occurs in financial markets. The square root, (Heston 1993) and SABR models are types of stochastic volatility models. The Heston model is one factor volatility model. This model is one of the more popular stochastic volatility models. The parameters of the Heston model are not wide enough to match the correct level of market volatility. This means that, the Heston model can generate sharp smiles or flat smiles, but not both simultaneously. Therefore, it is possible to add a parameter to do that. Another way to solve this problem is by modeling the variance using a variable with higher dimensions. (Da Fonseca et al. 2006) and (Gouriéroux 2006) replaced the variance of the Cox-Ingersoll-Ross with the Wishart process (Gauthier 2010). Their model offers relevant control of covariance dynamics as an n-dimensional matrix. Their model enables one to have a tighter control of the covariance dynamics as it is represented by a matrix of size n.

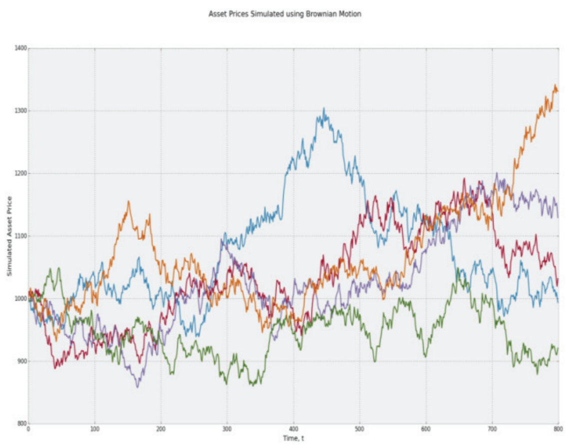
In Peter Christoffersen (2009) et al. extended the Heston model and named it the Double Heston model. The variance process of this extension is modeled with two uncorrelated processes. This model is more accurate than the original Heston model, yielding a comparable computing time. Experimental observations have shown that adding a third factor will yield little improvement (Pierre Gauthier 2010).

In this study, we have used the multi-level Monte Carlo method and antithetic variates as a variance reduction technique and then, we studied the Heston model with its two factors using the MLMC method.

2. Brownian Motion

The molecular movement pedestal is also known as the Brownian motion, This involves the random movement of microscopic particles suspended in a liquid or gas and are caused by collisions between these particles and molecules. This movement is named after its founder, Scottish botanist Robert Brown (1773-1858). Brown observed that pollen grains (suspended in water) seemed to move around the liquid at random. This intrigued him and he began to study about why this was happening, and tried to establish what force was driving these random fluctuations and changes in direction. He was not sure what was causing the motion, so he set out to rule out other possible causes. The key finding of Brown was that he proved that the movement was not due to the live pollen propelling itself, through his scrutiny of dead pollen grains and rock dust. He also noted that these smaller particles underwent a larger amount of vigorous movement and fluctuation. Contrary to popular belief, although Brown was the first to observe and document the phenomenon, he was unsure as to why it was happening. Later studies began to uncover that the Brownian movement was due to the buffeting of individual molecules in the water. Although pollen grains are 10,000 times larger than the water molecules, the cumulative effect of all that buffeting was strong enough to move the grains around. This is what resulted in the jerky and unpredictable movement of pollen grains.

Whilst, instinctively, you would think that random movement of pollen grains would be alike 1 in all directions and that the molecules would cancel each other out, that is in fact impossible. There will always be a slightly stronger grain that pushes one way more than another.



2.1. Definition

The stochastic process $W = (W_t), t \geq 0$ is called a Brownian motion or Wiener process if:

a) $W_0 = 0$.

b) W has independent increments, i.e. $W_t - W_s$ is independent of $W_t' - W_s'$ for all $0 \leq s' < t' \leq s < t < \infty$.

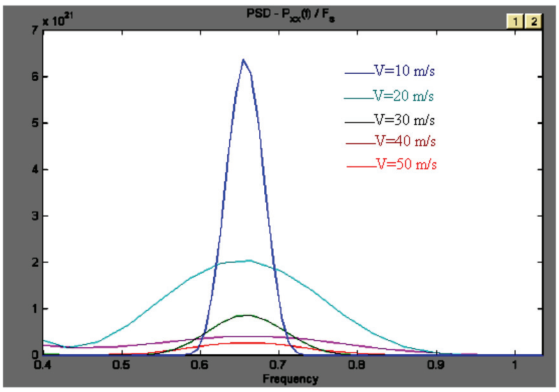
c) W has stationary increments, i.e. the distribution of $W_{t+u} - W_t$ only depends on u for $u \geq 0$.

d) Under Q, the increments $W_{t+u} - W_t$ of W are normally distributed with mean 0 and variance u, i.e. $W_{t+u} - W_t \sim \mathcal{N}(0, u)$.

e) W is time-continuous.

2.2. The impact of random waves and floating rate in bending

According to randomness of the ocean waves, dynamic reactions due to the effect of the waves are random with regard to floating. Since a ship on a river coming across random waves can have variable speeds, the dynamic reactions of that speed can vary in relation to other speeds. This will be further clarified when the range of bending moments (bends) in the middle of floating are at different speeds and can be plotted by the Mat lab's software mentioned range shown below:



The figure also indicates that when there is an increase in the floating rate, the energy of the bending moment (bend) decreases, but the frequency range increases.

2.3. Problem Definition

Suppose that we want to estimate $P = \mathbb{E}(\hat{P})$, such that $\hat{P}$ is a function of a sequence of random variables $(S_j)_{j \in J}$ on the same probability space $(\Omega, \mathcal{F}, \mathbb{P})$,

$$P = P(S_1, S, \dots, S_m).$$

If $|m| < \infty$, then

$$P = \mathbb{E}(\hat{P}) = \int_\Omega \dots \int_\Omega P(s_1, s_2, \dots, s_m)\, ds_1 \dots ds_m$$

Let us assume we know the payoff function $\hat{P}$, but we do not know anything about the stock price process S. For our European option, which only depends on the value of the asset at maturity T, this then reduces to the computation to a one-dimensional integral

$$P = \int_\Omega P(s_T)\, ds_T$$

In the following sections we want to solve the above equation using Monte Carlo simulation methods.

3. Monte Carlo Simulation

The Monte Carlo simulation, first suggested by (P. B. Boyle 1997) utilizes pseudorandom numbers in order to provide a path simulation of the price. This method is applicable to option pricing. We must obtain a sample path for the Monte Carlo simulation. The Black–Scholes model explains the stock price evaluation by way of the following stochastic differential equation :

$$dS_t = rS_t dt + \sigma S_t dW_t \qquad (1)$$

where the parameter r and σ are the mean rate of return, the volatility of the stock price, respectively and W_t is a standard Brownian motion. If we take the rate of return to be the same as the interest rate r, under the risk-neutral measure, we have the solution of Equation (1) as follows :

$$S(T) = S(0) \exp\left(\left(r - \frac{\sigma^2}{2}\right)T + \sigma W(T)\right) \qquad (2)$$

The sample path was generated for m periods through dividing the interval [0; T] to $m\ \delta t$ in order to generate a sample path of

$$S(t_j) = S(0) \exp\left(\left(r - \frac{\sigma^2}{2}\right)\delta t + \sigma\sqrt{\delta t} Z_j\right),$$

$$Z_j \sim N(0,1), \qquad j = 1,2, \dots, m$$

Option price, $X(\omega)$, for a European call option is

$max\ (S(T) - k, 0)$,

Where k is the strike price. To determine the sample mean of the discounted payoffs, we use n stock price paths and apply the risk free rate, r, to obtain

$X = \frac{1}{n}\sum_{i=1}^{n} X(\omega_i)\, e^{-r\delta t}$.

4. Antithetic Variables

With regard to financial pricing problems, the antithetic variates technique is the one of simplest and most widely used approaches. We regard the European call option problem on no-dividend stock. There is no need to evaluate this price by simulation, because the analytical solution of this problem exists, but this example is very applicable to introduce the antithetic variates method to pricing stocks. The stock price In the Black-Scholes model follows a log-normal diffusion. Under the risk-neutral measure we can generate Independent replications of the terminal stock price from the formula

$$S(t_j) = S(0)\exp\left(\left(r - \frac{\sigma^2}{2}\right)\delta t + \sigma\sqrt{\delta t}Z_j\right),$$

$$Z_j \sim N(0,1)\ , \qquad j = 1,2,\dots,m \qquad (3)$$

An unbiased estimator for pricing an option with n replications is given by

$$X = \frac{1}{n}\sum_{i=1}^{n} X(\omega_i)\, e^{-r\delta t}$$

In this study, we assume that Z_j has a standard normal distribution. In terms of the antithetic variates method if Z_j has a standard normal distribution, then also $-Z_j$ has standard normal distribution. The price $S(t_j)$ obtained from (3) with Z_j replaced by replacing $-Z_j$ instead of Z_j in Eq. (3) we obtain a valid sample from the terminal stock price distribution. Correspondingly, each

$$\tilde{X}(\omega_i) = e^{-r\delta t}\, max\left(\tilde{S}(t_j) - k, 0\right).$$

The option price is an unbiased estimator, therefore,

$$\hat{X}_{AV} = \frac{1}{n}\sum_{i=1}^{n}\frac{X_i + \tilde{X}_i}{2}$$

Since X_i and $\tilde{X}_i$ have the same variance,

$$Var\left[\frac{X_i + \tilde{X}_i}{2}\right] = \frac{1}{2}\left(Var[X_i] + cov[X_i, \tilde{X}_i]\right)$$

To increase efficiency, we require $2Var[\hat{X}_{AV}] \le Var[X]$. Thus, we require that

$$cov[X_i, \tilde{X}_i] \le 0.$$

5. Multi-level Monte Carlo

Note that Giles (M. B. Giles 2008) has introduced a multi-level Monte Carlo in option pricing. The Multi-level Monte Carlo approach is a type of Monte Carlo simulation that has a number of time steps of size $h_l = M^{-l}T$ that are different on each level $l = 0,1,\dots,L$, for integer $M \ge 2$, with the smallest time step h_L corresponding to the original h, which determines the size of the Euler discretization bias. For example, when $l = 0$, there is only one time step of size $h_0 = T$. When $l = 1$, there are two times steps each of size $h_1 = \frac{T}{M}$. Finally, when $l = L$, there are M^L times steps each of size $h_L = M^{-L}T$.

Let P denote the accurate derivative price

$$P = \mathbb{E}[\hat{P}_L(S)] = \mathbb{E}[\hat{P}_0(S)] + \sum_{l=1}^{L}\mathbb{E}[\hat{P}_l(S) - \hat{P}_{l-1}(S)]$$

where $\hat{P}_l(S)$ denotes the approximation to P on level l. Let $\hat{Y}_0$ be an estimator for $\mathbb{E}[\hat{P}_0(S)]$ using N_0 samples, and let $\hat{Y}_l$ for $l > 0$ be an estimator for $\mathbb{E}[\hat{P}_l(S) - \hat{P}_{l-1}(S)]$ using N_l paths. The simplest estimator that one might use is a mean of N_l independent samples, which for $l > 0$ is

$$\hat{Y}_l = \frac{1}{N_l}\sum_{i=1}^{N_l}[\hat{P}_l(s_i) - \hat{P}_{l-1}(s_i)]$$

The key point here is that the quantity $\hat{P}_l(s_i) - \hat{P}_{l-1}(s_i)$ comes from two discrete approximations with different time steps but the same Brownian path. The multi-level Monte Carlo estimator is given by

$$\hat{Y} = \sum_{l=0}^{L}\hat{Y}_l$$

The variance of the combined multi-level Monte Carlo estimator on level l is given by

$$Var[\hat{Y}_l] = Var\left[\frac{1}{N_l}\sum_{i=0}^{N_l}[\hat{P}_l(s_i) - \hat{P}_{l-1}(s_i)]\right]$$

$$= \frac{1}{{N_l}^2}\sum_{i=0}^{N_l} Var[\hat{P}_l(s_i) - \hat{P}_{l-1}(s_i)]$$

$$= \frac{V_l}{N_l}$$

Thus, the variance of the combined multi-level

Monte Carlo estimator is

$$Var[\hat{Y}] = \sum_{l=0}^{L} Var[\hat{Y}_l] = \sum_{l=0}^{L} \frac{V_l}{N_l} ,$$

where V_l = Var$[\hat{P}_l(S) - \hat{P}_{l-1}(S)]$. Furthermore,

V_l = Var$[\hat{P}_l(S) - \hat{P}_{l-1}(S)]$ =

Var$[\hat{P}_l(S)] + Var[\hat{P}_{l-1}(S)] - 2cov[\hat{P}_l(S), \hat{P}_{l-1}(S)]$.

So, the higher correlation between $\hat{P}_l(S)$ and $\hat{P}_{l-1}(S)$ is the lower variance of the multi-level Monte Carlo estimator.

By theorem 3.1 in Giles (2008), $Var[\hat{P}] \leq \frac{\epsilon^2}{2}$, where ϵ is a user-specifed accuracy.

The optimal number of sample paths for level l, in order to minimize the variance of the multi-level Monte Carlo estimator for a given computational cost C, is

$$N_l = \left[2\epsilon^{-2}\sqrt{V_l h_l}\left(\sum_{l=0}^{L}\sqrt{\frac{V_l}{h_l}}\right)\right].$$

Overall, Monte Carlo has computational cost proportional to ϵ^{-3}, whereas that of the multi-level Monte Carlo is proportional to $\epsilon^{-2}(\log \epsilon)^2$ due to reduced variance.

6. Stochastic Volatility Model

Volatility smile (smirk) has been generated by modern stochastic volatility models. Concerning the Black-Scholes model, the volatility smile indicates that out-of-the-money index puts have high prices. Also, in reaction to changes in relation to risk, these models illustrate how the volatility smile moves up and down. However, the data shows that the level and the slope of the smile fluctuations are largely independent. The One-factor stochastic volatility models can only show the smile of the slope (Peter Christoffersen 2009). These models cannot illustrate great independent fluctuations in its slope and level over time. We suggest the multi-level Monte Carlo Simulation for solving multi-factor stochastic volatility models to be used in this case.

6.1. One-Factor Volatility Model

The stochastic volatility models of Hull and White (1988), Melino and Turnbull (1990), and Heston (1993) allow for nonzero correlation between the stock variance and the level of the stock return. The helpfulness of stochastic volatility models in modeling the smile have been documented in several papers. At a given volatility level, a one-factor stochastic volatility model can generate steep smiles or flat smiles, but cannot generate them together for a given parameterization. The original Heston (1993) model, that is a one-factor stochastic volatility model, usually cannot fit the implied volatility smile (smirk) very well, particularly at short maturities. In order to compare the one and two factor stochastic volatility models, we use the one-factor Heston (1993) model that is one of the most common models in the literature of option pricing. Then, in the next section we improve the Heston MATLAB code for the two-factor stochastic volatility model. Let at time *t* the underlying asset follow the diffusion (Heston 1993).

$$dS_t = rS_t dt + \sqrt{v_t S_t dw_t^1}, \qquad (4)$$

where r is constant interest rate, v_t is the volatility at time *t* and w_t^1 is a Wiener process . If the volatility follows an Ornstein–Uhlenbeck process [e.g., used by Stein and Stein (1991)]

$$d\sqrt{v_t} = -\beta\sqrt{v_t d_t} + \sigma dw_t^2 \qquad (5)$$

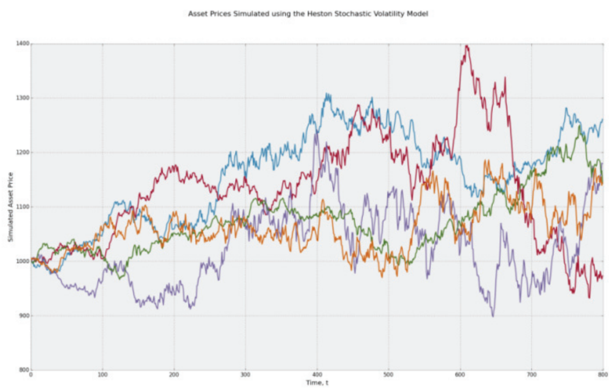
using Ito's lemma, we show that the variance process is as follows

$$dv_t = [\sigma^2 - 2\beta v_t]d_t + 2\sigma\sqrt{v_t}dw_t^2 \qquad (6)$$

This can be written as the familiar square-root process [used by Cox, Ingersoll, and Ross (1985)]

$$dv_t = \kappa[\theta - v_t]d_t + \sigma\sqrt{v_t}dw_t^2 \qquad (7)$$

where w_t^2 has correlation ρ with w_t^1, κ is mean reversion, θ is long term volatility and σ is the volatility of volatility.



6.2. Two-Factor Volatility Model

The Two-Factor Volatility (Pierre Gauthier 2010) model suggested by (Peter Christoffersen 2009) et al. introduces a second factor for the variance. Using the multi-level Monte Carlo method, we show that the two-factor models have much more flexibility in ruling the slope and the level of the smile. Also, the other benefit of two-factor models is that they have more flexibility when compared to the volatility term structure. We change the volatility of volatility of first and second factors (σ_1, σ_2) and fix other parameters.

Suppose that we use the process of the dividend stock price instead of the variance of the risk-neutral. Then this process is caused by two factors

$$dS_t = rS_t dt + \sqrt{v_t^1}\ S_t dw_t^1 + \sqrt{v_t^2}\ S_t dw_t^2 \qquad (8)$$

$$dv_t^1 = \kappa_1[\theta_1 - v_t^1]d_t + \sigma_1\sqrt{v_t^1}dw_t^3 \qquad (9)$$

$$dv_t^2 = \kappa_2[\theta_2 - v_t^2]d_t + \sigma_2\sqrt{v_t^2}dw_t^4 \qquad (10)$$

The following correlation structure is specified

$$\mathbb{E}[dw_t^1\ dw_t^3] = \rho_1 d_t$$

$$\mathbb{E}[dw_t^2\ dw_t^4] = \rho_2 d_t$$

$$\mathbb{E}[dw_t^1\ dw_t^2] = \mathbb{E}[dw_t^3\ dw_t^4]$$

$$= \mathbb{E}[dw_t^1\ dw_t^4] = \mathbb{E}[dw_t^2\ dw_t^3] = 0$$

The two-factor volatility (Pierre Gauthier 2010) model provides more flexibility in modeling the level of volatility. When the correlation between the returns and their variance $(\rho) \approx 0$, we have the symmetric smile, but when ρ moves to ±1 then we have a highly asymmetric smile, with $\rho \approx -1$ corresponding to a negative slope and $\rho \approx +1$ to a positive slope. In the one-factor Heston model, ρ is constant over maturities. It means that when the slope of the smile varies substantially across maturities, the one-factor Heston model has trouble giving an sufficient fit to market implied volatilities, although when the slopes are all relatively flat or all relatively steep it does a good job. Adding a second factor of volatility allows for two different correlations and, thus, for two different regimes of volatility.

7. Numerical Results

7.1. Example 1

We presented the multi-level Monte Carlo method in option pricing under the geometric Brownian motion. First, we used a scheme in the Monte Carlo simulation and then compared these results with the multi-level Monte Carlo and antithetic variables technique .Finally, we obtain results and showed the efficiency of the multi-level Monte Carlo approach. In the first example, we simulated the value of a European option with $r = 0.05, \sigma = 0.3, T = 1\,(year), S_0 = 1, K = 0.7$, and $NRepl$ is the number of independent replications. The reference value, calculated analytically is

$$V \cong 0.3440$$

Table 1: The simulation results with parameters:

$r = 0.05, \sigma = 0.3, T = 1\,(year), S_0 = 1, K = 0.7$, and $NRepl = 1000000$

Type of estimator	Simulated Value	Std. Error
MLMC	0.343998	1.394353 e-009
AVMC	0.343954	8.703768 e-005
NMC	0.343733	2.933359 e-004

Table 2: Comparison of the simulations by MLMC method for different values of strike price with parameters:

$r = 0.05, \sigma = 0.3, T = 1\,(year), S_0 = 0.5$, and $NRepl = 100000$

	K=0.3	K=0.4	K=0.45	K=0.5
	Analytical value	Simulated Value	Simulated Value	Simulated Value
	0.2160	0.1323	0.0985	0.0712
	Simulated value	Simulated Value	Simulated Value	Simulated Value
Type of estimator	Error	Error	Error	Error

MLMC	0.216450 3.742365 e-008	0.132801 3.101688 e-008	0.098839 2.582735 e-008	0.071387 2.023090 e-008
AVMC	0.2159 1.1682 e-004	0.1325 1.6330 e-004	0.0986 1.8602 e-004	0.0713 1.9509 e-004
NMC	0.2158 1.5121 e-004	0.1328 4.3923 e-004	0.0985 4.0128 e-004	0.0713 3.5747 e-004

From this example it is clear that the multi-level Monte Carlo method has a smaller error margin when compared to the antithetic variable technique and standard Monte Carlo method.

7.2. Example 2

We solve Two-Factor Stochastic Volatility model under the European call option using the Multi Level Monte Carlo method with $L = 4, M = 4, N_l = 2000000$, and present results:

$$dS_t = rS_t dt + \sqrt{v_t^1}\ S_t dw_t^1 + \sqrt{v_t^2}\ S_t dw_t^2$$

$$dv_t^1 = \kappa_1[\theta_1 - v_t^1]d_t + \sigma_1\sqrt{v_t^1}dw_t^3$$

$$dv_t^2 = \kappa_2[\theta_2 - v_t^2]d_t + \sigma_2\sqrt{v_t^2}dw_t^4$$

$r = 0.05, v_0^1 = 0.01, v_0^2 = 0.03, \kappa_1 = 4,$

$\kappa_2 = 4,\ \theta_1 = 0.03,\ \theta_2 = 0.01,\ \sigma_1 = 0.25,$

$\sigma_2 = 0.25, \rho_1 = -0.80, \rho_2 = -0.80,$

$T = 1\ (year),\ S_0 = 1,\ K = 1.$

The reference value, calculated analytically is:

value = 0.104741

***Remark** : the chosen data have the following meaning: The parameter T represents the maturity of the option, the initial price S_0 and the strike price are 1, The risk-free short rate (r) is set to 5%, the initial volatility (v_0^1, v_0^2) are (1%,3%), the voltility of volatility (σ_1, σ_2) are (25%,25%), the mean reversion (κ_1, κ_2) are (400%,400%), the correlation (ρ_1, ρ_2) are (80%,80%) and long term volatility (θ_1, θ_2) are (3%,1%).*

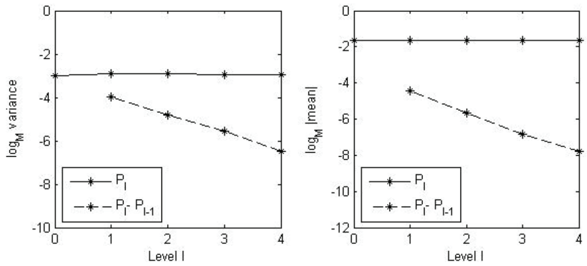


Figure 1: Comparison of the variance (left figure) and the mean (right figure) of the MLMC and the MC method over different levels under the European call option.

In Figure 1 we compared the variance and the mean of the MLMC method and the standard Monte Carlo method without any variance reduction technique under the European call option by using an Euler discretization scheme. To produce the two plots, $2 * 10^6$ sample paths have been generated. The left plot shows the logarithm with base *M* of the variance of P_l , the discrete approximation of the variable *P* using the time step size $h_l = M^{-l}T$, and P_{l-1} ,respectively, against the number of levels *l.* the variance of P_l, used for the standard MC method, is more or less constant whereas the variance of $P_l - P_{l-1}$, used for the MLMC method, decreases as *l* increases. At level $l = 4$, $var(P_l - P_{l-1})$ is approximately 4^4 times smaller than $var(P_l)$.

The right plot of Figure 1 represents the logarithm with base *M* of the absolute value of the mean of P_l and $P_l - P_{l-1}$, respectively, against the number of levels *l*. At level *l* = 4, the absolute value of $\mathbb{E}[P_l - P_{l-1}]$, used for the MLMC method, is about 4^6 times smaller than the absolute value of $\mathbb{E}[P_l]$, used for the standard Monte Carlo method.

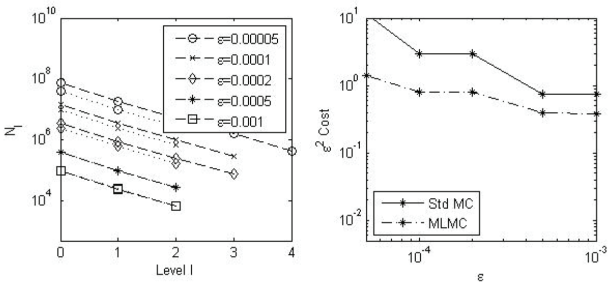


Figure 2: Left hand picture shows the number of paths per level at each target accuracy. Right hand picture shows the computation time, scaled by ϵ^2.

Figure 2: Output from the multi-level Monte Carlo code made available by (M. B. Giles 2008). The left hand picture shows the number of paths per level at each target accuracy. Right hand picture shows the computation time, scaled by ϵ^2. Above pictures are for a European call option.

We change the volatility of volatility of first and second factors (σ_1, σ_2) and fix other parameteres.

$$(\sigma_1, \sigma_2) = (10\%, 10\%)$$

value = 0.104783

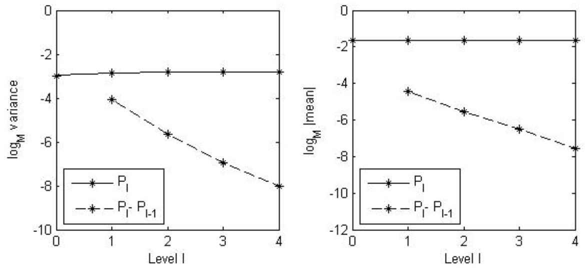


Figure 3: Comparison of the variance (left figure) and the mean (right figure) of the MLMC and the MC method over different levels under the European call option.

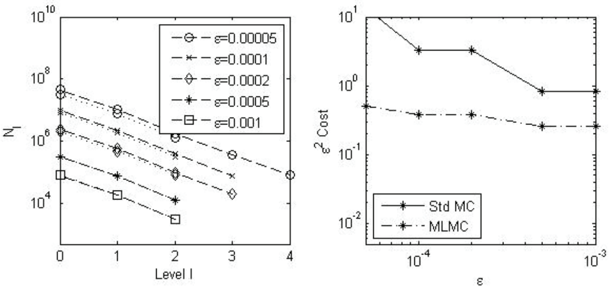


Figure 4: Left hand picture shows the number of paths per level at each target accuracy. Right hand picture shows the computation time, scaled by ϵ^2.

$$(\sigma_1, \sigma_2) = (10\%, 25\%)$$

value = 0.104658

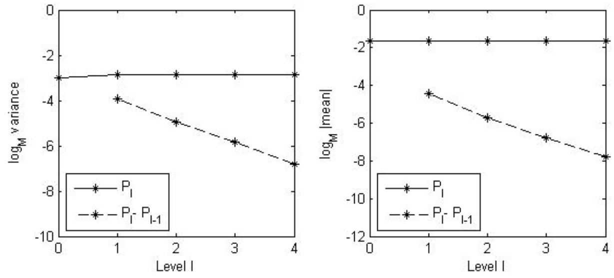


Figure 5: Comparison of the variance (left figure) and the mean (right figure) of the MLMC and the MC method over different levels under the European call option.

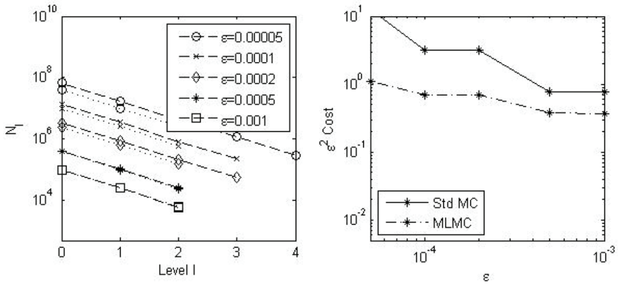


Figure 6: Left hand picture shows the number of paths per level at each target accuracy. Right hand picture shows the computation time, scaled by ϵ^2.

$$(\sigma_1, \sigma_2) = (50\%, 25\%)$$

value = 0.104040

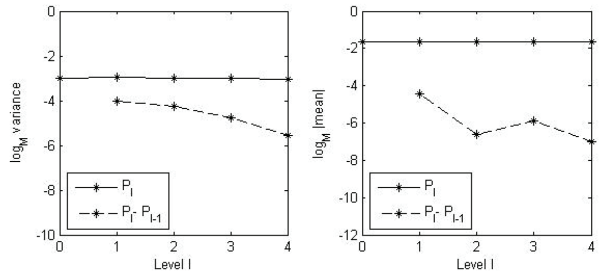


Figure 7: Comparison of the variance (left figure) and the mean (right figure) of the MLMC and the MC method over different levels under the European call option.

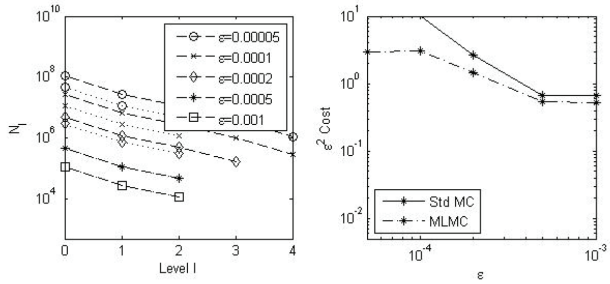


Figure 8: Left hand picture shows the number of paths per level at each target accuracy. Right hand picture shows the computation time, scaled by ϵ^2.

8. Conclusion

In this paper, we showed that the multi-level Monte Carlo simulation provides better error reduction in pricing the European call option when compared with the standard Monte Carlo and antithetic variables techniques. We rewrite the MATLAB code of the multi-level Monte Carlo method for the two-factor Stochastic volatility model and compare it with the one-factor Stochastic volatility model. The numerical results indicate that the computational cost is also substantially reduced.

We believe we are going to see more over the next year or so. It is related to financial rises and the reverberations of that huge shockwave and more intervention by governments and central bankers than ever before. Quantitative easing, not just in the U.S. but in the Eurozone and in Asia, has played a major role in creating more uncertainty about what is happening. When you have the Fed purchasing US$85 billion of bonds every month and trying to decide when to take its foot off the accelerator and when to raise rates, that type of uncertainty is categorically contributing to the volatility of volatility. Therefore, we change the volatility of volatility first and second factors (σ_1, σ_2) and fix other parameteres. When (σ_1, σ_2) increases the total variance increases also. We leave open the question as to whether additional factors are needed, and how they would improve pricing performance.

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There is no conflict of interest for all authors.